Repeated Matching Games, an Empirical Framework

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Introduction

Dynamic aspects are crucial for matching games

- In labor economics
- In family economics
- In mergers and acquisitions
- In school choice

In all these contexts, matching today affects agents state variables and therefore future matching prospects:

- Labor: human capital acquisition
- Marriage: fertility, moving and career decisions

Motivation

We develop a framework for these dynamic matching problems

- With and without unobserved heterogeneity
- With finite or infinite (stationary) horizon
- With equilibrium prediction, structural estimation and comparative statics

In which agents are forward looking and account for how matches today affect future matches

Dynamic Matching

Our model generalizes static matching *à la* Choo and Siow (2006) using state variable transition in the spirit of Rust (1987)

- One-to-one matching, transferable utility
- Solution concept: competitive equilibrium
- Matching market clears each period
- Matching affects evolution of agents' state variable
- Agents maximize present value of profits
- No information asymmetry
- No friction unless explicitly modelled

Related Literature

- Static, one-to-one, transferable utility models: Choo and Siow (2006), Fox et al. (2018), Chiappori et al. (2017), Galichon and Salanie (2020)
- Dynamic discrete choice: Rust (1987)
- Search and matching: Shimer and Smith (2000), Eeckhout (2001), Peski (2021), Ederer (2021)
- Dynamic matching close to our work: Erlinger et al. (2015), McCann et al. (2015), Choo (2015)

Outline

1. Baseline model
2. Model with unobserved heterogeneity
3. Stationary equilibrium
4. Methods for computing the stationary equilibrium
5. Estimation
6. Application: location switching costs of Swedish engineers

Baseline Model

Workers & Firms

Infinte horizon model: time subscript are dropped

- Continuum of workers of type $x \in \mathcal{X}$ (also called state variable)
- Finite number of types. Each type is possibly multidimensional
- Total mass of workers N
- Similarly for firms: $y \in \mathcal{Y}$, total mass M
- Options available to type x worker: $\mathcal{Y}_0 = \mathcal{Y} \cup \{0\}$
- Options available to type y firm: $\mathcal{X}_0 = \mathcal{X} \cup \{0\}$

Type Transitions

Types evolve as function of current match (x, y)

- $P_{x'|xy} = P(x'|x, y)$: transition mass function for worker state if worker x matches with firm y
- $Q_{y'|xy} = Q(y'|x, y)$ transition mass function for firm state if firm y matches with worker x
- $\sum_{x'} P_{x'|xy} = \sum_{y'} Q_{y'|xy} = 1$ (for now)
- Transitions are deterministic if $P_{x'|xy}$ and $Q_{y'|xy}$ are 0 or 1.

Matches and Transfers

- μ_{xy}: mass of matches between types x, y
- μ_{x0}, μ_{0y} masses of unmatched x and y
- w_{xy}: monetary transfer paid by y tp x
- Flow profit of worker x matched to y:

$$\alpha_{xy} + w_{xy}$$

- Flow profit of firm y matched to x:

$$\gamma_{xy} - w_{xy}$$

- Flow profit of unmatched firms and workers: α_{x0} and γ_{y0}

Bellman Equations

Constant aggregate state at beginning of period: n, m

$\rightarrow w$, μ are functions of n, m.

Transition to next period aggregate state:

$$\sum_{x,y} P_{x'|xy}\mu_{xy} = m'_{x'} \text{ and } \sum_{x,y} Q_{y'|xy}\mu_{xy} = n'_{y'}$$

Worker's Bellman equation:

$$U_x(m,n) = \max_{y\in\mathcal{Y}_0}\left\{\alpha_{xy} + w_{xy}(m,n) + \beta\sum_{x'\in\mathcal{X}} P_{x'|xy}U_{x'}(P\mu, Q\mu)\right\}$$

Fims Bellman equation:

$$V_y(m,n) = \max_{x\in\mathcal{X}_0}\left\{\gamma_{xy} - w_{xy}(m,n) + \beta\sum_{y'\in\mathcal{Y}} Q_{y'|xy}V_{y'}(P\mu, Q\mu)\right\}$$

U and V are worker's and firm's lifetime utilities

Competitive Equilibrium

A competitive equilibrium is a pair μ, w such that if $\mu_{xy} > 0$

$$y \in \arg\max_{\tilde{y} \in \mathcal{Y}_0} \left\{ \alpha_{x\tilde{y}} + w_{x\tilde{y}}\left(m, n\right) + \beta \sum_{x' \in \mathcal{X}} P_{x'|x\tilde{y}} U_{x'}\left(P\mu, Q\mu\right) \right\}$$

$$x \in \arg\max_{\tilde{x} \in \mathcal{X}_0} \left\{ \gamma_{\tilde{x}y} - w_{\tilde{x}y}\left(m, n\right) + \beta \sum_{y' \in \mathcal{Y}} Q_{y'|\tilde{x}y} V_{y'}\left(P\mu, Q\mu\right) \right\}$$

Primal Problem

Define total match surplus

$$\Phi_{xy} = \alpha_{xy} + \gamma_{xy}$$

Theorem

The matching policy μ *maximizes the* **social planner's Bellman**

$$W(m,n) = \max_{\mu_{xy} \geq 0} \left\{ \sum_{xy \in \mathcal{X}_0 \mathcal{Y}_0} \mu_{xy} \Phi_{xy} + \beta W(P\mu, Q\mu) \right\}$$

subject to the constraints $\sum_{y \in \mathcal{Y}_0} \mu_{xy} = m_x$ *and* $\sum_{x \in \mathcal{X}_0} \mu_{xy} = n_y$

Consequence: A competitive equilibrium exists and the economy sum of profits $W(m,n)$ is uniquely determined

Dual Problem

Make use of linear duality theory to directly compute lifetime utilities U and V

$$D(m,n) = \min_{U,V} \left\{ \sum_{x \in \mathcal{X}} m_x U_X(m,n) + \sum_{y \in \mathcal{Y}} n_y V_y(m,n) + \beta D(P\mu, Q\mu) \right\}$$

Subject to constraints

$$U_x(m,n) + V_y(m,n) \geq$$
$$\Phi_{xy} + \beta \sum_{x' \in \mathcal{X}} P_{x'|xy} U_{x'}(P\mu, Q\mu) + \beta \sum_{y' \in \mathcal{Y}} Q_{y'|xy} V_{y'}(P\mu, Q\mu)$$

Then recover equilibrium transfers $w(m,n)$

Unobserved Heterogeneity

Econometric Unobservables

In data, agents of same x match with many different y's $\rightarrow$ **introduce econometric errors.**

Worker i's flow profit:

$$\alpha_{xy} + w_{xy} + \epsilon_{iy}$$

Firm j's flow profit:

$$\gamma_{xy} + w_{xy} - \eta_{jx}$$

Assumption

- *(ϵ_{iy}) and (η_{jx}) are independent over time, as in Rust (1987)*
- *(ϵ_{iy}) and $(\epsilon_{i'y})$, (η_{jx}) and $(\eta_{j'x})$, (ϵ_{iy}) and (η_{jx}) are iid*
- *(ϵ_{iy}) and (η_{jx}) are distributed as extreme value 1, as in Choo and Siow (2006)*

Regularized Primal Problem

Social planner Bellman equation now writes

$$W(m,n) = \max_{\mu_{xy}} \left\{ \sum_{xy \in \mathcal{X}_0 \mathcal{Y}_0} \mu_{xy} \Phi_{xy} - \mathcal{E}(\mu; m, n) + \beta W(P\mu, Q\mu) \right\}$$

Where $\mathcal{E}(\mu; m, n)$ is the entropy. Under Gumbel shocks, it writes

$$\mathcal{E}(\mu; m, n) = \sum_{xy} \mu_{xy} \log \mu_{xy} + \sum_{x} n_x \log n_x + \sum_{y} m_y \log m_y$$

Regularized Dual Problem

$$D(m,n) = \min_{U,V} \sum_{x\in\mathcal{X}} m_x U_X(m,n) + \sum_{y\in\mathcal{Y}} n_y V_y(m,n)$$

$$+\sum_{x,y} n_x m_y \exp\left(\Phi_{xy} - U_x(m,n) - V_y(m,n) + \beta \sum_{x'} P_{x'|xy} U_{x'}(P\mu, Q\mu) + \beta \sum_{y'} Q_{y'|xy} V_{y'}(P\mu, Q\mu)\right)$$
$$+ \beta D(P\mu, Q\mu)$$

Equilibrium matching is

$$\mu_{xy}(m,n) = \exp\Big(\Phi_{xy} - U_x(m,n) - V_y(m,n)$$
$$+ \beta \sum_{x'} P_{x'|xy} U_{x'}(P\mu, Q\mu) + \beta \sum_{y'} Q_{y'|xy} V_{y'}(P\mu, Q\mu)\Big)$$

Stationary Equilibrium

Constant Aggregate State

Constant aggregate state is n, m that satisfy for a given equilibrium

$$m = P\mu(m, n) \text{ and } n = Q\mu(m, n) \tag{1}$$

i.e. (m, m) remains next period's state if current state is (m, n). Competitive μ is then called the **stationary equilibrium**

Theorem

For every $M, N > 0$, a constant aggregate state exists such that $\sum_x m_x = M$ and $\sum_y n_y = N$

Proof shows goes through showing that (1) defines a contraction for (m, n) under logit shocks assumption (using Brouwer theorem)

Computing the Stationary Equilibrium

We could compute the entire equilibrium for all aggregate states and then search for constant aggregate state, but instead we develop two much faster methods:

- Mathematical Programming with Equilibrium Constraints (Su and Judd (2012)).
  - Easy to implement
  - Very fast with small number of types
- Rewrite the steady state conditions as an optimization problem and apply Chambolle-Pock algorithm (Chambolle and Pock (2010))
  - Can deal with large numbers of types

MPEC

Rewrite steady state as system of equations on n, m, U, V:

$$\sum_x \mu_{xy} = m_x \text{ and } \sum_y \mu_{xy} = n_y$$

$$\sum_{xy} P_{x'|xy}\mu_{xy} = m_x \text{ and } \sum_{xy} Q_{y'|xy}\mu_{xy} = n_y$$

$$\mu_{xy} = n_x m_y \exp\left(\Phi_{xy} - U_x - V_y + \beta \sum_{x'} P_{x'|xy} U_{x'} + \beta \sum_{y'} Q_{y'|xy} V_{y'}\right)$$

In practise: define the problem in JuMP (a Julia package), and solve using IPOPT

Steady State as an Optimization Problem

Define the following:

$$Z(m,n,U,V,U',V',\beta) = \sum_{xy} \exp\left(\Phi_{xy} - U_x - V_y + \beta \sum_{x'} P_{x'|xy} U'_{x'} + \beta \sum_{y'} Q_{y'|xy} V'_{y'} \right) - \sum_x m_x - \sum_y n_y$$

Assume $\beta = 1$, and set $F(m,n,U,V) = Z(m,n,U,V,U,V)$, then the following optimization problem

$$\min_{U,V} \max_{m,n} F(m,n,U,V,1)$$

has first order conditions

$$\text{In } m,n\text{: } \sum_x \mu_{xy} = m_x \text{ and } \sum_y \mu_{xy} = n_y$$

$$\text{In } U,V\text{: } \sum_{xy} P_{x'|xy}\mu_{xy} - \sum_x \mu_{xy} = 0 \text{ and } \sum_{xy} Q_{y'|xy}\mu_{xy} - \sum_y \mu_{xy} = 0$$

Chambolle-Pock

Chambolle and Pock (2010) for min max problems: choose a step ϵ and do:

$$\begin{cases} \tilde{m}^t = 2m^t - m^{t-1}, \tilde{n}^t = 2n^t - n^{t-1} \\ U^{t+1} = U^t - \epsilon \partial_U F(\tilde{m}^t, \tilde{m}^t, U^t, V^t) \\ V^{t+1} = V^t - \epsilon \partial_V F(\tilde{m}^t, \tilde{m}^t, U^t, V^t) \\ m^{t+1} = m^t + \epsilon \partial_m F(m^t, n^t, U^{t+1}, V^{t+1}) \\ n^{t+1} = n^t + \epsilon \partial_n F(m^t, n^t, U^{t+1}, V^{t+1}) \end{cases}$$

Then the algorithm converges to a solution to $\min_{U,V} \max_{m,n} F(m, n, U, V, 1)$

$\beta < 1$

If $\beta < 1$, constant aggregate state does not rewrite as an optimization problem because then

$$\partial_U F = \beta \sum_{xy} P_{x'|xy}\mu_{xy} - \sum_x \mu_{xy} \text{ and } \partial_V F = \beta \sum_{xy} Q_{y'|xy}\mu_{xy} - \sum_y \mu_{xy}$$

But if we run the analog to Chambolle-Pock it still converges !

$$\begin{cases} \tilde{m}^t = 2m^t - m^{t-1}, \tilde{n}^t = 2n^t - n^{t-1} \\ U^{t+1} = U^t - \epsilon\left(\beta \partial_U Z(\tilde{m}^t, \tilde{m}^t, U^t, V^t, U^t, V^t) + \partial_{U'} Z(\tilde{m}^t, \tilde{m}^t, U^t, V^t, U^t, V^t)\right) \\ V^{t+1} = V^t - \epsilon\left(\beta \partial_V Z(\tilde{m}^t, \tilde{m}^t, U^t, V^t, U^t, V^t) + \partial_{V'} Z(\tilde{m}^t, \tilde{m}^t, U^t, V^t, U^t, V^t)\right) \\ m^{t+1} = m^t + \epsilon \partial_m Z(\tilde{m}^t, \tilde{m}^t, U^t, V^t, U^t, V^t) \\ n^{t+1} = n^t + \epsilon \partial_n Z(\tilde{m}^t, \tilde{m}^t, U^t, V^t, U^t, V^t) \end{cases}$$

Extensions

- All results and methods go through with unmatched agents
- Add an incoming flow of agents, as well as retiring agents (P and Q non stochastic)
- Add normalization variable to both MPEC and Chambolle-Pock to match total masses N and M

Estimation

Strategy

- Assume data comes from constant aggregate state
- Observations on (x,x',y,y'): worker and firm state in current and next period, matching in current period $\tilde{\mu}$
- we want to estimate transition matrices P and Q as well as parameters λ that parametrize total surplus Φ^{λ}, where $\Phi^{\lambda}_{xy} = \sum_k \phi^k_{xy} \lambda^k$
- As in Rust (1987), transition matrices can be estimated in first stage, by counting number of agents who transition from x to x' and fro y to y'
- Both MPEC and Chambolle-Pock can be adapted to estimation of λ

Estimation with MPEC

Sample log likleihood is

$$\sum_{xy} \tilde{\mu}_{xy} \log \mu_{xy}(\lambda)$$

Or if we observe singles:

$$2\sum_{xy} \tilde{\mu}_{xy} \log \mu_{xy}(\lambda) + \sum_{x0} \tilde{\mu}_{x0} \log \mu_{x0}(\lambda) + \sum_{0y} \tilde{\mu}_{0y} \log \mu_{0y}(\lambda)$$

We simply need to maximize the log likelihood, subject to MPEC constraint

Estimation with Chambolle-Pock

Modify Z to include λ

$$Z(m,n,U,V,U',V',\lambda,\beta)=\sum_{xy}\exp\left(\Phi^{\lambda}_{xy}-U_x-V_y+\beta\sum_{x'}P_{x'|xy}U'_{x'}+\beta\sum_{y'}Q_{y'|xy}V'_{y'}\right)$$
$$-\sum_x m_x-\sum_y n_y-\sum_{xy}\tilde{\mu}_{xy}\Phi^{\lambda}_{xy}$$

Then $\partial_\lambda F=0$ is a moment matching condition $\sum_{x,y}\mu_{xy}\phi^k_{xy}=\sum_{x,y}\tilde{\mu}_{xy}\phi^k_{xy}$
The same intuitions as before are valid for optimization:

$$\min_{U,V,\lambda}\max_{m,n}H(m,n,U,V,\lambda,\beta)$$

Method Comparison

Table 1: Speed comparison (in seconds) - Equilibrium computation

	3 types	10 types	30 types
Chambolle-Pock	.15	2.62	160.61
MPEC	.02	1.03	-

Table 2: Speed comparison (in seconds) - Estimation

	2 parameters	10 parameters
Chambolle-Pock	3.03	76.70
MPEC	1.36	3.92

10 types

Application: Geographic Mobility of Swedish Engineers

We use this framework to estimate location switching cost by age group for Swedish engineers from 1970-1990 (Fox (2010)). Parametrization is the following:

$$x = \{previous\_location, age\_group\} \text{ and } y = \{location\}$$

With 5 age groups and 4 locations. Surplus is

$$\Phi^{\lambda}_{xy} = \sum_{k=1}^{5} \lambda^k \mathbb{1}_{[x_{age}=k]} dist_{xy}$$

Where $dist_{xy}$ is the distance in kilometers between x's region and y's region'

Transition Matrices Estimates

We assume a 10% probability of aging, i.e going from age group k to age group $k+1$, and compute attrition rates for workers and firms directly from the data. Transition matrices are then estimated as:

$$P_{x'|xy} = \begin{cases} (1-\rho)\delta_x \tilde{\mu}_{xy} \text{ if } x_{age} = x'_{age} \text{ and } y_{loc} = x'_{loc} \\ \rho\delta_x \tilde{\mu}_{xy} \text{ if } x_{age} + 1 = x'_{age} \text{ and } y_{loc} = x'_{loc} \\ 0 \text{ otherwise} \end{cases}$$

$$Q_{y'|xy} = \begin{cases} \delta_y \tilde{\mu}_{xy} \text{ if } y_{loc} = y'_{loc} \\ 0 \text{ otherwise} \end{cases}$$

Choose $\beta = .95$

Parameters Estimate

Table 3: Estimates for moving cost by age bin

	λ_1	λ_2	λ_3	λ_4	λ_5
Chambolle-Pock	-51.81	-49.86	-49.10	-47.28	-52.97
MPEC	-48.42	-48.88	-51.22	-48.31	-50.32

In euros/kilometers

Conclusion

We introduce a concept of repeated matching games:

- All agents match in each period
- Market clears every period
- Agents' type evolve according to current match
- Steady state exists for any given total masses of agents

Still on the agenda:

- Identification issues a la Kalouptsidi et al. (2021), Kalouptsidi et al. (2019)
- Theoretical convergence of Chambolle-Pock when $\beta < 1$
- Conditions for uniqueness of steady state
- Computation of standard errors

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